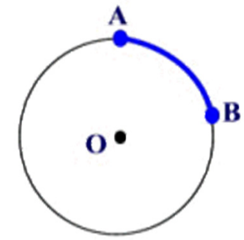
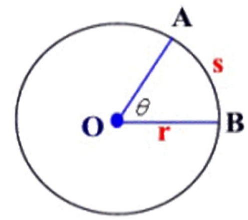
§4.1 RADIAN MEASURE

KEY CONCEPTS



- An **arc** of a circle is a "portion" of the circumference of the circle.
- The **length of an arc** is simply the length of its "portion" of the circumference. Actually, the circumference itself can be considered an arc length.
- The length of an arc (or arc length) is traditionally symbolized by **s**.



- In the diagram at the right, it can be said that "$\overset{\frown}{AB}$ *subtends* angle θ".
  Definition: ***subtend*** *- to be opposite to*
- The **radian measure** θ of a central angle of a circle is defined as the ratio of the length of the arc the angle subtends, **s**, divided by the radius of the circle, ***r***.

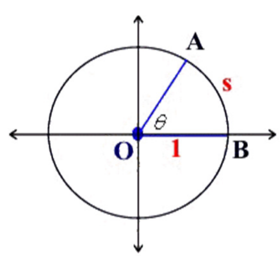
$$\theta = \frac{s}{r} = \frac{\text{length of subtended arc}}{\text{length of radius}}$$

From this definition we can obtain:

RADIANS	**DEGREES**
Arc length of a circle: $arc\ length = \theta r$ $s = \theta r$	Arc length of a circle: $arc\ length = \theta \bullet \frac{\pi}{180} \bullet r$

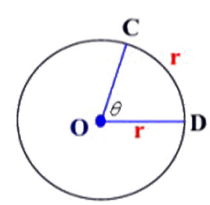
- When working in the unit circle, with radius 1, the length of the arc equals the radian measure of the angle.

$$\underbrace{\theta = \frac{s}{r} = \frac{s}{1} = s}_{\theta = s}$$



- **A radian** is the measure of an angle θ that, when drawn as a central angle, subtends an arc whose length equals the length of the radius of the circle.

$$\theta = \frac{s}{r} = \frac{r}{r} = 1$$



$\angle COD$ = **1** radian

- **Relationship between Degrees and Radians:**
  When the arc length equals an entire circumference, we can use $s = \theta r$ to get $2\pi r = \theta r$, and $2\pi = \theta$. This implies that $2\pi = 360°$.

So, $1° = \frac{\pi}{180}$ radians and $1 \text{ radian} = \frac{180}{\pi}$ degrees

$$\boxed{1 \text{ radian} = \frac{180°}{\pi} = 57.296°}$$

To change from degrees to radians, multiply by $\frac{\Pi}{180°}$	To change from radians to degrees, multiply by $\frac{180°}{\Pi}$

- The time rate of change of angle θ by a rotating body is the **angular velocity**, written ω (omega). It is measured in radians/second.

  If *v* is the **linear velocity** (in m/s) and *r* is the radius of the circle (in m), then $v = r\omega$.

  Note: If *r* is in cm, *v* will be in cm/s.

Examples:

1. Convert 50° to radians.

2. Convert $\frac{\Pi}{6}$ to degrees.

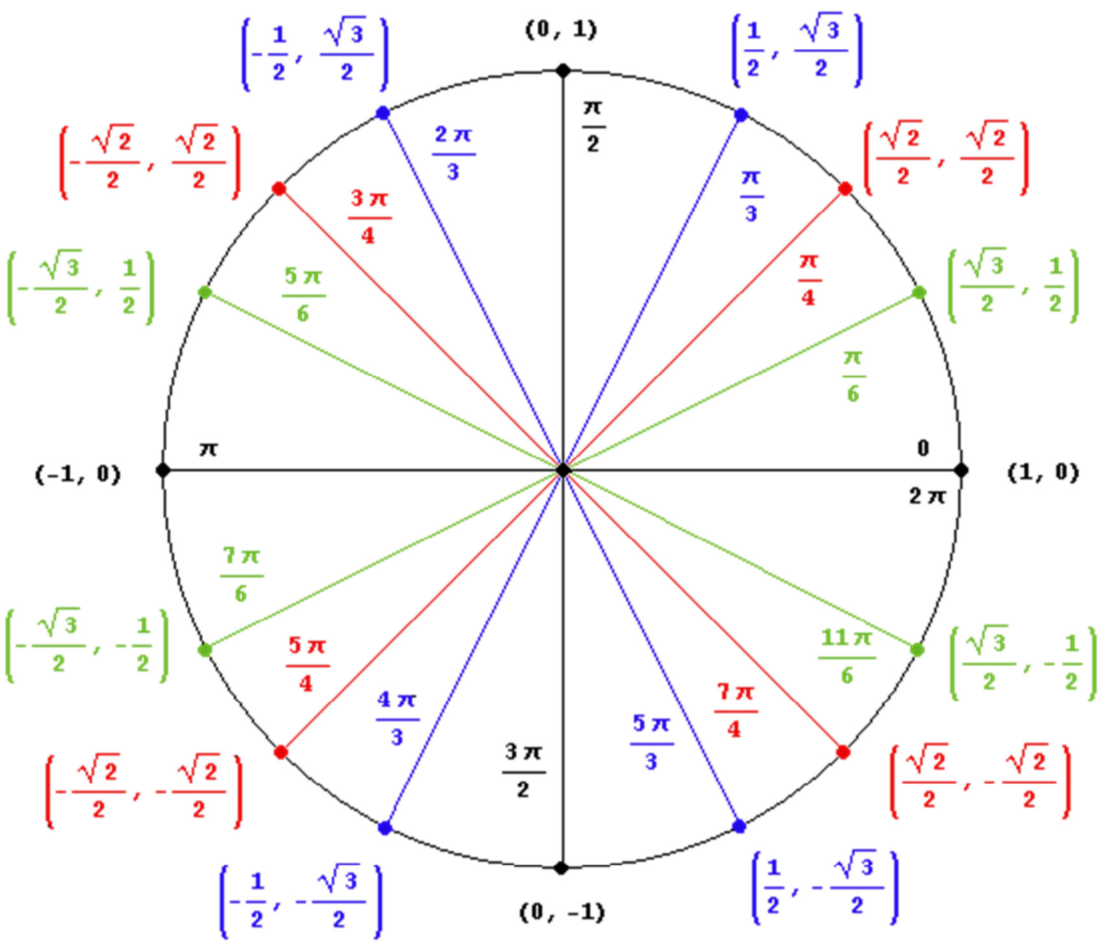
3. How long is the arc subtended by an angle of $\frac{7\Pi}{4}$ radians on a circle of radius 20 cm?

4. A bicycle with tires 90 cm in diameter is travelling at 25 km/h. What is the angular velocity of the tire in radians per second?

§4.2 TRIGONOMETRIC RATIOS & SPECIAL ANGLES

KEY CONCEPTS

- Special angles between 0 and 2∏ together with their sine and cosine are displayed on a unit circle. These special angles may be useful in solving trigonometry problems.



§4.3 EQUIVALENT TRIGONOMETRIC EXPRESSIONS

KEY CONCEPTS

- **Equivalent trigonometric expressions** are expressions that yield the same value for all values of the variable.
- An **identity** is an equation that is true for all values of the variable for which the expressions on both sides of the equation are defined.
- An identity involving trigonometric expressions is called a **trigonometric identity**.

Trigonometric Identities Featuring $\frac{\Pi}{2}$			
Cofunction Identities		**More Identities Involving $\frac{\Pi}{2}$**	
$\sin x = \cos\left(\frac{\Pi}{2} - x\right)$	$\cos x = \sin\left(\frac{\Pi}{2} - x\right)$	$\sin\left(x + \frac{\Pi}{2}\right) = \cos x$	$\cos\left(x + \frac{\Pi}{2}\right) = -\sin x$
$\tan x = \cot\left(\frac{\Pi}{2} - x\right)$	$\cot x = \tan\left(\frac{\Pi}{2} - x\right)$	$\tan\left(x + \frac{\Pi}{2}\right) = -\cot x$	$\cot\left(x + \frac{\Pi}{2}\right) = -\tan x$
$\csc x = \sec\left(\frac{\Pi}{2} - x\right)$	$\sec x = \csc\left(\frac{\Pi}{2} - x\right)$	$\csc\left(x + \frac{\Pi}{2}\right) = \sec x$	$\sec\left(x + \frac{\Pi}{2}\right) = -\csc x$

Examples:

1. Given that $\cot\frac{2\Pi}{7} \approx 0.7975$, use equivalent expressions to evaluate the following to four decimal places.

a) $\tan\frac{3\Pi}{14}$ **b)** $\tan\frac{11\Pi}{14}$

§4.4 COMPOUND ANGLE FORMULAS

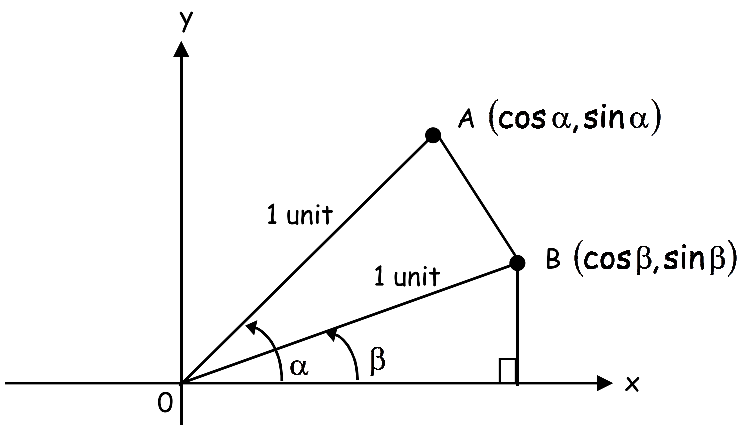
KEY CONCEPTS

- A trigonometric expression that depends on two or more angles is known as a **compound angle expression**.

- **Subtraction Formula (Cosine)**

  cos (A - B) = cos A cos B + sin A sin B

  Proof:



We will use this diagram to help prove the subtraction formula.

In general, we will compute c^2 in two ways (using the Law of Cosines and also using the distance formula) and compare the two results.

- **Addition Formula (Cosine)**

  cos (A + B) = cos A cos B - sin A sin B

  Proof:

  cos (A - B) = cos A cos B + sin A sin B } ***subtraction formula for Cosine***
  cos (A – (-B)) = cos A cos (-B) - sin A sin (-B) } ***replacing b with -b***
  = cos A cos B + sin A sin B } ***negative angles***

- **Addition Formula (Sine)**

  sin (A + B) = sin A cos B + cos A sin B

  Proof:

  Recall the cofunction identities $\sin x = \cos\left(\frac{\Pi}{2} - x\right)$ and $\cos x = \sin\left(\frac{\Pi}{2} - x\right)$.

  Apply these and the subtraction formula for the cosine.

  $\sin(A+B) = \cos\left(\frac{\Pi}{2} - (A+B)\right)$ } ***apply a cofunction idenity***

  $= \cos\left((\frac{\Pi}{2} - A) - B)\right)$ } ***regroup the terms in the brackets***

  $= \cos\left(\frac{\Pi}{2} - A\right)\cos B + \sin\left(\frac{\Pi}{2} - A\right)\sin B$ } ***apply subtraction formula for Cosine***

  $= \sin A \cos B + \cos A \sin B$

- **Subtraction Formula (Sine)**

  sin (A - B) = sin A cos B - cos A sin B

  Proof:

  sin (A + B) = sin A cos B + cos A sin B } ***addition formula for Sine***
  sin (A – (-B)) = sin A cos (-B) + cos A sin (-B) } ***replacing B with -B***
  = sin A cos B - cos A sin B } ***negative angles***

Example:

1. Use an appropriate compound angle formula to determine an exact value for $\cos\left(\frac{5\Pi}{12}\right)$.

2. Use the appropriate compound angle formula to express the following as a single trigonometric function, and then determine an exact value for it.
$\sin\frac{\Pi}{2}\cos\frac{\Pi}{4}-\cos\frac{\Pi}{2}\sin\frac{\Pi}{4}$

3. Angles x and y are located in the first quadrant such that $\sin y=\frac{24}{25}$ and $\cos x=\frac{3}{5}$.
Determine an exact value for $sin(x-y)$.

§4.5 PROVE TRIGONOMETRIC IDENTITIES

KEY CONCEPTS

- In this section, you will use the following basic trigonometric identities to prove other identities.

Pythagorean Identity

$\cos^2 \alpha + \sin^2 \alpha = 1$

Quotient Identity

$\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha}$

Reciprocal Identities

$\csc\alpha = \dfrac{1}{\sin\alpha}$ $\sec\alpha = \dfrac{1}{\cos\alpha}$ $\cot\alpha = \dfrac{1}{\tan\alpha}$

Compound Angle Formulas

$\sin(A - B) = \sin A \cos B - \cos A \sin B$
$\sin(A + B) = \sin A \cos B + \cos A \sin B$
$\cos(A - B) = \cos A \cos B + \sin A \sin B$
$\cos(A + B) = \cos A \cos B - \sin A \sin B$

Double Angle Identities

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta\end{aligned}$$

Examples:

1. Prove $\dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} - \dfrac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} = 2\tan 2\theta$.

2. Prove $\frac{1-\sin 2\theta}{\cos 2\theta}=\frac{\cos 2\theta}{1+\sin 2\theta}$.